SINGLE CHANGEPOINT MLE:

Obs = $x_1, x_2, \dots ; x_n$ for $t = 1, 2 \dots n$

Params: Emissions = θ_1, θ_2

Changepoint = τ

Modded as:

State weight = $w(t) = \dfrac{1}{1 + e^{-(t-\tau)/s}}$ w/ s = smoothing parameter.

$\theta(t) = [1 - w(t)]\theta_1 + w(t)\theta_2$

Emission over time

Assuming temporal independence of timepoints given parameters

$$\mathcal{L}(\theta_1, \theta_2, \tau, s) = \prod_{t=1}^{n} f(X_t \mid \theta(t))$$

$$\ln \mathcal{L} = \ell(\theta_1, \theta_2, \tau, s) = \ln \prod_t f(X_t \mid \theta(t)) = \sum_t \ln f(X_t \mid \theta(t))$$

Using poisson likelihood: $f(X_t \mid \theta(t)) = \dfrac{\theta(t)^{X_t} e^{-\theta(t)}}{X_t!}$

$$\ell = \sum_t \ln \frac{\theta(t)^{X_t} e^{-\theta(t)}}{X_t!} = \sum_t \left[\ln \theta(t)^{X_t} + \ln e^{-\theta(t)} - \ln(X_t!) \right] = \sum_t \left[X_t \ln \theta(t) - \theta(t) - \underbrace{\ln(X_t!)}_{\text{constant}} \right]$$

$$\ell \propto \sum_t \left[X_t \ln \theta(t) - \theta(t) \right]$$

Start MLE:

$$\frac{\partial \ell}{\partial \tau} = \frac{\partial}{\partial \tau} \sum_t \left[X_t \ln \theta(t) - \theta(t) \right] = \sum_t \left[X_t \frac{\partial}{\partial \tau} \ln \theta(t) - \frac{\partial}{\partial \tau} \theta(t) \right]$$

$$\frac{\partial}{\partial\tau}\theta(t) = \frac{\partial}{\partial\tau}\Big[[1-\omega(t)]\theta_1 + \omega(t)\,\theta_2\Big] = \frac{\partial}{\partial\tau}\Big[\cancelto{0}{\theta_1} - \theta_1\omega(t) + \theta_2\,\omega(t)\Big] = [\theta_2-\theta_1]\frac{\partial}{\partial\tau}\omega(t)$$

$$\frac{\partial}{\partial\tau}\omega(t) = \frac{\partial}{\partial\tau}\left[1+e^{-\frac{(t-\tau)}{s}}\right]^{-1} = -1\left[1+e^{-\frac{(t-\tau)}{s}}\right]^{-2}\underbrace{\frac{\partial}{\partial\tau}\left[1+e^{-\frac{(t-\tau)}{s}}\right]}$$

$$\frac{\partial}{\partial\tau}\left[\cancelto{0}{1}+e^{-\frac{(t-\tau)}{s}}\right] = e^{-\frac{(t-\tau)}{s}}\frac{\partial}{\partial\tau}\left[-\frac{(t-\tau)}{s}\right] \; ; \; -\frac{\partial}{\partial\tau}\frac{t-\tau}{s} = \frac{1}{s}\frac{\partial}{\partial\tau}\, t-\tau = -\frac{1}{s}$$

$$= \frac{-e^{-\frac{(t-\tau)}{s}}}{s}$$

$$\frac{\partial}{\partial\tau}\omega(t) = -1\left[1+e^{-\frac{(t-\tau)}{s}}\right]^{-2}\left[-\frac{1}{s}e^{-\frac{(t-\tau)}{s}}\right] = \frac{e^{-\frac{(t-\tau)}{s}}}{s\left[1+e^{-\frac{(t-\tau)}{s}}\right]^2} = \frac{e^{-\frac{(t-\tau)}{s}}+1}{s\left[1+e^{-\frac{(t-\tau)}{s}}\right]^2} - \frac{1}{s\left[1+e^{-\frac{(t-\tau)}{s}}\right]^2}$$

$$= \frac{1}{s\left[1+e^{-\frac{(t-\tau)}{s}}\right]} - \frac{1}{s\left[1+e^{-\frac{(t-\tau)}{s}}\right]^2} = \frac{1}{s}\left[\underbrace{\frac{1}{\left[1+e^{-\frac{(t-\tau)}{s}}\right]}}_{\omega} - \underbrace{\frac{1}{\left[1+e^{-\frac{(t-\tau)}{s}}\right]^2}}_{\omega^2}\right] = \frac{1}{s}\left[\frac{1}{u}-\frac{1}{u^2}\right]$$

lol

$$u = 1+e^{-\frac{(t-\tau)}{s}} \Rightarrow \frac{1}{s}\left[\frac{u^2-u}{u^3}\right] = \frac{1}{s}\left[\frac{u-1}{u^2}\right] = \frac{1}{s}\left[\frac{1}{\omega}-1\right]\omega^2 = \frac{1}{s}\left[\omega-\omega^2\right] = \frac{1}{s}\cdot\omega(1-\omega)$$

$$\omega = \frac{1}{u}, \quad u = \frac{1}{\omega}$$

$$\boxed{\frac{\partial}{\partial\tau}\omega(t) = \frac{\omega(1-\omega)}{s}}, \quad \frac{\partial\theta(t)}{\partial\tau} = [\theta_2-\theta_1]\frac{\partial}{\partial\tau}\omega(t) = \boxed{[\theta_2-\theta_1]\cdot\frac{\omega(t)(1-\omega(t))}{s} = \frac{\partial}{\partial\tau}\theta(t)}$$

$$\frac{\partial}{\partial\tau}\ln\theta(t) = \frac{\frac{\partial}{\partial\tau}\theta(t)}{\theta(t)} = \frac{[\theta_2-\theta_1]\,\omega(t)\,(1-\omega(t))}{s}\cdot\frac{1}{\underbrace{[1-\omega(t)]\theta_1+\omega(t)\,\theta_2}_{\theta_1-\theta_1\omega(t)+\theta_2\omega(t)\;=\;\theta_1+[\theta_2-\theta_1]\omega(t)}} = \frac{[\theta_2-\theta_1]\omega(t)(1-\omega(t))}{s\left[\theta_1+[\theta_2-\theta_1]\omega(t)\right]}$$

$$\frac{\partial \ell}{\partial \tau} = \sum_t \left[X_t \frac{\partial}{\partial \tau} \ln \theta(t) - \frac{\partial}{\partial \tau} \theta(t) \right] = \sum_t \left[X_t \frac{[\theta_2 - \theta_1]\omega(t)(1-\omega(t))}{s[\theta_1 + [\theta_2-\theta_1]\omega(t)]} - \frac{[\theta_2-\theta_1]\omega(t)(1-\omega(t))}{s} \right]$$

$$\cancel{\Rightarrow \sum_t \left[X_t [\theta_2 - \theta_1]\omega(t)(1-\omega(t)) - \right.}$$

Alternatively: $\frac{\partial \ell}{\partial \tau} = \frac{\partial}{\partial \tau} \sum_t [X_t \ln \theta(t) - \theta(t)] = \sum_t \frac{\partial \ell}{\partial \theta} \cdot \frac{\partial \theta}{\partial \omega} \cdot \frac{\partial \omega}{\partial \tau}$

$$\frac{\partial \ell}{\partial \theta} = \frac{\partial}{\partial \theta}[X_t \ln \theta - \theta] = X_t \frac{\partial}{\partial \theta} \ln \theta - \frac{\partial}{\partial \theta}\theta = X_t \frac{1}{\theta} - 1 = \frac{X_t}{\theta} - 1$$

$$\frac{\partial \theta}{\partial \omega} = \frac{\partial}{\partial \omega}[1-\omega]\theta_1 + \omega\theta_2 = \frac{\partial}{\partial \omega}[\theta_1 - \theta_1\omega + \theta_2\omega] = \theta_2 - \theta_1$$

$$\frac{\partial \omega}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{1}{1+e^{-(t-\tau)/s}} = \frac{\omega(1-\omega)}{s} \quad (\text{from pg. 2})$$

$$\boxed{\frac{\partial \ell}{\partial \tau} = \sum_t \left(\frac{X_t}{\theta(t)} - 1\right)(\theta_2 - \theta_1)\left[\frac{\omega(1-\omega)}{s}\right]}$$

Expand to confirm equivalence, ⟶ or not ... too much work

$$\Rightarrow \sum_t \left(\frac{X_t - \theta}{\theta}\right)(\theta_2 - \theta_1)\left[\frac{\omega(1-\omega)}{s}\right] = \sum_t \left[\quad \ldots\ldots \right.$$

$$\frac{\partial \ell}{\partial \theta_1} = \sum_t \frac{\partial \ell}{\partial \theta(t)} \cdot \frac{\partial \theta(t)}{\partial \theta_1} \quad , \quad \frac{\partial \theta(t)}{\partial \theta_1} = \frac{\partial}{\partial \theta_1} [1-\omega]\theta_1 + \omega \theta_2 = \boxed{1-\omega = \frac{\partial \theta(t)}{\partial \theta_1}}$$

$$\boxed{\frac{\partial \ell}{\partial \theta_1} = \sum_t \left(\frac{x_t}{\theta_t} - 1\right)(1-\omega(t))}$$

$$\frac{\partial \theta(t)}{\partial \theta_2} = \frac{\partial}{\partial \theta_2} [1-\omega]\theta_1 + \omega \theta_2 = \boxed{\omega = \frac{\partial \theta(t)}{\partial \theta_2}}$$

$$\frac{\partial \ell}{\partial \theta_2} = \sum_t \frac{\partial \ell}{\partial \theta(t)} \cdot \frac{\partial \theta(t)}{\partial \theta_2} = \boxed{\sum_t \left(\frac{x_t}{\theta_t} - 1\right)\omega = \frac{\partial \ell}{\partial \theta_2}}$$

$$\omega = \frac{1}{1+e^{-\frac{(t-\tau)}{s}}} \quad , \quad z = e^{-\frac{(t-\tau)}{s}}$$

$$\cancel{\omega = \frac{1}{1+e^z}}$$

$$\omega = \frac{1}{1+z} \quad , \quad \cancel{\omega + \omega z} \Rightarrow 1+z = \frac{1}{\omega}$$

$$z = \frac{1}{\omega} - \frac{1}{1} = \frac{1-\omega}{\omega} = \frac{(1-\omega)\omega}{\omega^2}$$

$$\frac{\partial \ell}{\partial s} = \sum_t \underbrace{\frac{\partial \ell}{\partial \theta(t)} \cdot \frac{\partial \theta(t)}{\partial \omega(t)}}_{\text{Already have}} \cdot \frac{\partial \omega(t)}{\partial s} \quad , \quad \frac{\partial \omega(t)}{\partial s} = \frac{\partial}{\partial s}\Big[\underbrace{1 + e^{-(t-\tau)/s}}_{u}\Big]^{-1} = -\underbrace{\left[1+e^{-\frac{(t-\tau)}{s}}\right]^{-2}}_{\omega^2} \frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial s}\left[\overset{0}{\cancel{1}} + e^{-\frac{(t-\tau)}{s}}\right] = \cancel{\left[\frac{(t-\tau)}{s}\right]} e^{-\frac{(t-\tau)}{s}} \frac{\partial}{\partial s} -(t-\tau)s^{-1} = e^{-\frac{(t-\tau)}{s}}\left[-(t-\tau)s^{-2}\right] = \frac{-(t-\tau)e^{-\frac{(t-\tau)}{s}}}{s^2}$$

$$\frac{\partial \omega(t)}{\partial s} = -\left[1+e^{-\frac{(t-\tau)}{s}}\right]^{-2}\left[\frac{-(t-\tau)}{s^2} e^{-\frac{(t-\tau)}{s}}\right] = \frac{t-\tau}{s^2} \cdot \omega^2 \cdot \underbrace{e^{-\frac{(t-\tau)}{s}}}_{z} = \frac{t-\tau}{s^2} \cdot \omega^2 \cdot \frac{(1-\omega)\omega}{\omega^2} = \left(\frac{t-\tau}{s^2}\right)(1-\omega)\omega$$

$$\boxed{\frac{\partial \omega(t)}{\partial s} = \left(\frac{t-\tau}{s^2}\right)\omega(1-\omega)}$$

$$\frac{\partial \ell}{\partial s} = \sum_t \frac{\partial \ell}{\partial \theta(t)} \cdot \frac{\partial \theta(t)}{\partial \omega(t)} \cdot \frac{\partial \omega(t)}{\partial s} = \boxed{\sum_t \left(\frac{x_t}{\theta(t)} - 1\right)(\theta_2 - \theta_1)\left[\left(\frac{t-\tau}{s^2}\right)\omega(1-\omega)\right] = \frac{\partial \ell}{\partial s}}$$