RESCORLA-WAGNER MODEL

$\Delta V_A^{(t)} = V_A^{(t+1)} - V_A^{(t)}$ ⇒ The change in value for cue A paired w/ a reward

Model claim: $\Delta V_A^{(t)} = \alpha(\lambda^{(t)} - V_A^{(t)})$, for $\lambda^{(T)}$ = actual outcome

⇒ Update is proportional to prediction error. $V_A^{(t)}$ = current expectation for cue A.

α = learning rate.

⇒ $\Delta V_A^{(t)} = \alpha[\lambda^{(t)} - V_A^{(t)}]$

$V_A^{(t+1)} - V_A^{(t)} = \alpha[\lambda^{(t)} - V_A^{(t)}]$ ⇒ $V_A^{(t+1)} = \alpha\lambda^{(t)} + V_A^{(t)}(1-\alpha)$

⇒ Future reward is weigated average of previous reward and previous expectation.

Maximum Likelihood Estimation:

$y^{(t)}$ = choice of participant $\in \{0, 1\}$ on trial t

$\begin{array}{cc} \downarrow & \downarrow \\ \text{choice A} & \text{B} \end{array}$

$P(y^{(t)} = 1) = p^{(t)}$ = softmax rule #

$P(y^{(t)}) = [p^{(t)}]^{y^{(t)}}[1 - p^{(t)}]^{1-y^{(t)}}$

The probability of $y^{(t)}$ is conditionally independent of previous trials given current state i.e. Markov Assumption

$$\mathcal{L} = \prod^{T} \underbrace{[p^{(t)}]^{y^{(t)}} [1-p^{(t)}]^{1-y^{(t)}}}_{z}$$

$$\ln \mathcal{L} = \ell = \ln \prod^{T} z = \sum^{T} \ln z = \sum^{T} \left[y^{(t)} \ln p^{(t)} + (1-y^{(t)}) \ln (1-p^{(t)}) \right] \rightarrow$$ Binary Cross Entropy loss

for $$p^{(t)} = \frac{e^{\beta_c V_A^{(t)}}}{e^{\beta_c V_A(t)} + e^{\beta_c V_B^{(t)}}} \Big\} \rightarrow \Sigma \Big\}$$ softmax

★ Move index to subscript for cleaner notation

$$\ln \mathcal{L} = \ell = \sum^{T} \left[y_t \ln p_t + (1-y_t) \ln (1-p_t) \right], \quad p_t = \frac{e^{\beta_c V_{A,t}}}{e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}}}$$

$$\frac{\partial \ell}{\partial \beta_c} = \sum^{T} \frac{\partial \ell}{\partial p_t} \frac{\partial p_t}{\partial \beta_c}, \quad \frac{\partial p_t}{\partial \beta_c} = \frac{\partial}{\partial \beta_c} \underbrace{e^{\beta_c V_{A,t}}}_{u} \underbrace{\left[e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}} \right]^{-1}}_{v} = \frac{\partial u}{\partial \beta_c} v + u \frac{\partial v}{\partial \beta_c}$$

$$\frac{\partial u}{\partial \beta_c} = V_{A,t} e^{\beta_c V_{A,t}}, \quad \frac{\partial v}{\partial \beta_c} = -1 \left[e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}} \right]^{-2} \left[V_{A,t} e^{\beta_c V_{A,t}} + V_{B,t} e^{\beta_c V_{B,t}} \right]$$

$$\frac{\partial p_t}{\partial \beta_c} = \underbrace{\frac{V_{A,t} e^{\beta_c V_{A,t}}}{e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}}}}_{V_{A,t} p_t} - \underbrace{\frac{e^{\beta_c V_{A,t}} \left[V_{A,t} e^{\beta_c V_{A,t}} + V_{B,t} e^{\beta_c V_{B,t}} \right]}{\left[e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}} \right]^2}} \Rightarrow V_{A,t} p_t - p_t^2 \frac{V_{A,t} e^{\beta_c V_{A,t}} + V_{B,t} e^{\beta_c V_{B,t}}}{e^{\beta_c V_{A,t}}}$$

$$\underbrace{\frac{e^{\beta_c V_{A,t}}}{e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}}}}_{p_t} \cdot \underbrace{\frac{\left[V_{A,t} e^{\beta_c V_{A,t}} + V_{B,t} e^{\beta_c V_{B,t}} \right]}{e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}}}}_{p_t \cdot V_{A,t} e^{\beta_c V_{A,t}} + V_{B,t} e^{\beta_c V_{B,t}} / e^{\beta_c V_{A,t}}}$$

(3)

$$\frac{\partial \ell}{\partial p_t} = \sum^T y_t \frac{\partial}{\partial p_t} \ln p_t + (1-y_t)\frac{\partial}{\partial p_t} \ln(1-p_t) = \sum^T \frac{y_t}{p_t} - \frac{(1-y_t)}{(1-p_t)}$$

$$\frac{\frac{\partial}{\partial p_t}(1-p_t)}{1-p_t}$$

☆ Simplify $\frac{\partial p_t}{\partial \beta_c}$ according to $p_t(1-p_t)\Delta V_t$ for $\Delta V_t = V_{A,t} - V_{B,t}$

$$\frac{\partial \ell}{\partial \beta_c} = \sum^T \left[\frac{y_t}{p_t} - \frac{(1-y_t)}{(1-p_t)}\right] \cdot p_t(1-p_t)\Delta V_t = \sum^T \left[\frac{y_t(1-p_t) - p_t(1-y_t)}{p_t(1-p_t)}\right] \cdot p_t(1-p_t)\Delta V_t$$

$$\frac{\partial \ell}{\partial \beta_c} \Rightarrow \sum^T \Delta V_t \left[y_t(1-p_t) - p_t(1-y_t)\right] = \sum^T \Delta V_t \left[y_t - y_t p_t - p_t - y_t p_t\right]$$

$$\boxed{\frac{\partial \ell}{\partial \beta_c} = \sum^T \Delta V_t [y_t - p_t]}$$

Next $\frac{\partial \ell}{\partial \alpha}$

$$\frac{\partial \ell}{\partial \alpha} = \sum^T \underbrace{\frac{\partial \ell}{\partial p_t}}_{\text{done}} \frac{\partial p_t}{\partial V_{A,t}} \frac{\partial V_{A,t}}{\partial \alpha} + \frac{\partial \ell}{\partial p_t} \frac{\partial p_t}{\partial V_{B,t}} \frac{\partial V_{B,t}}{\partial \alpha}$$

$$\frac{\partial V_{A,t}}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\alpha \lambda_t + \underbrace{(1-\alpha) V_{A,t-1}}_{\frac{\partial}{\partial \alpha}(1-\alpha) \cdot V_{A,t-1} + (1-\alpha)\frac{\partial}{\partial \alpha} V_{A,t-1}}\right] = \lambda_t - V_{A,t-1} + (1-\alpha)\frac{\partial V_{A,t-1}}{\partial \alpha},$$

☆ But note, if cue A is absent on trial $t-1$, then $\frac{\partial V_{A,t-1}}{\partial \alpha} = 0$

(4)

$$\frac{\partial p_t}{\partial V_{A,t}} = \frac{\partial}{\partial V_{A,t}} \frac{e^{\beta_c V_{A,t}}}{e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}}} := \frac{\partial}{\partial V_{A,t}} \frac{u}{v}$$

$$\frac{\partial}{\partial V_{A,t}} e^{\beta_c V_{A,t}} = \beta_c e^{\beta_c V_{A,t}} = u'$$

$$\frac{\partial}{\partial V_{A,t}} \Big[\underbrace{e^{\beta_c V_{A,t}}}_{a} + \underbrace{e^{\beta_c V_{B,t}}}_{b}\Big] = \beta_c e^{\beta_c V_{A,t}} = v'$$

$$\frac{\partial p_t}{\partial V_{A,t}} = \frac{u'v - uv'}{v^2} \quad \cancel{= \beta_c e^{\beta_c V_{A,t}} e^{\beta_c V_{A,t}} - e^{\beta_c V_{A,t}}}$$

$$= \frac{\beta_c e^{\beta_c V_{A,t}}\left[e^{\beta_c V_{A,t}} + e^{\beta_c V_{B,t}}\right] - e^{\beta_c V_{A,t}} \beta_c e^{\beta_c V_{A,t}}}{v^2}$$

$$\Rightarrow \frac{\beta_c e^{2\beta_c V_{A,t}} + \beta_c e^{\beta_c (V_{A,t} + V_{B,t})} - \beta_c e^{2\beta_c V_{A,t}}}{v^2} = \frac{\beta_c e^{\beta_c (V_{A,t} + V_{B,t})}}{v^2} = \underbrace{\frac{e^{\beta_c V_{A,t}}}{v}}_{p} \cdot \underbrace{\frac{e^{\beta_c V_{B,t}}}{v}}_{1-p} \cdot \beta_c$$

$$\cancel{v^2 = a^2 + b^2 - 2ab = e^{2\beta_c V_{A,t}} + e^{2\beta_c V_{B,t}} - 2e^{\beta_c (V_{A,t} + V_{B,t})}}$$

$$\boxed{\frac{\partial p_t}{\partial V_{A,t}} = p(1-p)\beta_c}$$

By symmetry w/ $\frac{\partial \ell}{\partial p_t}$, $\cancel{\frac{\partial p_t}{\partial p_t}}$

$$\boxed{\frac{\partial \ell}{\partial p_t} \cdot \frac{\partial p_t}{\partial V_{A,t}} = \sum^{T} \beta_c \left[y_t - p_t\right] = \frac{\partial \ell}{\partial V_{A,t}}}$$

Quotient rule

$$f = \frac{u}{v}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} u v^{-1}$$

$$= \frac{\partial u}{\partial x} v^{-1} + u \frac{\partial v^{-1}}{\partial x}$$

$\frac{\partial v^{-1}}{\partial x}$ for $v(x)$

$$\Rightarrow \frac{\partial v^{-1}}{\partial x} = -v^{-2} \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} v^{-1} - u v^{-2} \frac{\partial v}{\partial x}$$

$$= \frac{u'}{v} - \frac{uv'}{v^2}$$

$$= \frac{u'v - uv'}{v^2}$$

$$\frac{\partial p_t}{\partial v_{B,t}} = \frac{\partial}{\partial v_{B,t}} \frac{u}{v}, \quad \frac{\partial u}{\partial v_{B,t}} = \frac{\partial}{\partial v_{B,t}} e^{\beta_c v_{A,t}} = 0 = u', \quad \frac{\partial v}{\partial v_{B,t}} = \beta_c e^{\beta_c v_{B,t}}$$

$$\Rightarrow \frac{u'v - uv'}{v^2} = \frac{0 \cdot v' - e^{\beta_c v_{A,t}} \cdot \beta_c e^{\beta_c v_{B,t}}}{v^2} \xrightarrow[\text{w/ } \frac{\partial p_t}{\partial v_{A,t}}]{\text{By symmetry}} \boxed{-p(1-p)\beta_c = \frac{\partial p_t}{\partial v_{B,t}}}$$

$$\cancel{\frac{\partial \ell}{\partial \alpha} = \sum^T} \quad \boxed{\frac{\partial \ell}{\partial v_{B,t}} = \frac{\partial \ell}{\partial p_t} \cdot \frac{\partial p_t}{\partial v_{B,t}} \xrightarrow[\frac{\partial p_t}{\partial v_{A,t}}]{\text{By symmetry w/}} -\sum^T \beta_c [y_t - p_t]}$$

$$\frac{\partial \ell}{\partial \alpha} = \sum^T \frac{\partial \ell}{\partial v_{A,t}} \cdot \frac{\partial v_{A,t}}{\partial \alpha} + \frac{\partial \ell}{\partial v_{B,t}} \cdot \frac{\partial v_{B,t}}{\partial \alpha}$$

$$\boxed{\frac{\partial \ell}{\partial \alpha} = \sum^T \beta_c (y_t - p_t) \left[\frac{\partial v_{A,t}}{\partial \alpha} - \frac{\partial v_{B,t}}{\partial \alpha}\right]}$$