~~SINGLE CHANGEPOINT POISSON~~ - LAPLACE APPROXIMATION PROOF

We claim that the curvature around the MAP / MLE is related to the variance / uncertainty of the estimated parameters.

Let $f(\theta) = \ln p(\theta | X)$ and $\hat{\theta}$ = MAP/MLE vector

Taylor expansion of $f(\theta)$ @ $\hat{\theta}$.

$$f(x)_a = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots$$

$$f(\theta)_{\hat{\theta}} = f(\hat{\theta}) + f'(\hat{\theta})(\theta - \hat{\theta}) + \frac{f''(\hat{\theta})}{2!}(\theta - \hat{\theta})^2 + \cdots$$

$$= f(\hat{\theta}) + \nabla f(\hat{\theta})^T(\theta - \hat{\theta}) + \frac{\nabla^2 f(\hat{\theta})}{2}(\theta - \hat{\theta})^2 + \cdots$$

$$\Rightarrow f(\hat{\theta}) + \nabla f(\hat{\theta})^T(\theta - \hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^T \underbrace{H(\hat{\theta})}_{\text{Hessian} = \nabla^2 f(\hat{\theta})}(\theta - \hat{\theta})$$

but since $\hat{\theta}$ is a peak $\nabla f(\hat{\theta}) \approx 0$

$\therefore$ for ~~$O(2)$~~ $O(2)$ $f(\theta)_{\hat{\theta}} = f(\hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^T H(\hat{\theta})(\theta - \hat{\theta})$

~~since $f(\theta) = \ln p(\theta | X)$, $p(\theta | X) = \exp(f(\theta))$~~

Also note that $f(\hat{\theta})$ [log-~~likelihood~~ posterior @ $\hat{\theta}$] is just a constant

$\therefore$ $f(\theta)_{\hat{\theta}} = \frac{1}{2}(\theta - \hat{\theta})^T H(\hat{\theta})(\theta - \hat{\theta})$ + constant

Recall log MV Gaussian $= \ln\left[\underbrace{\frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}}}_{\text{constant}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)\right] = -\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) + \text{constant}.$

By matching forms of log-posterior @ MLE/MAP, that is $f(\theta)_{\hat{\theta}}$ is a multivariate gaussian w/ $x = \theta$, $\mu = \hat{\theta}$, $\boxed{\Sigma = -H(\hat{\theta})^{-1}}$