~~1D~~ Gaussian MLE 4/29/26 (1)

$\ln_e x = y$

$e^y = x$

$\ln_e[e^y] = y$

$x \sim \mathcal{N}(\mu, \Sigma)$

1D

$x_i \sim p(x_i | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

$\mathcal{L}(x) = \prod_{n=1}^{N} p(x_n | \mu, \sigma^2)$

$\log \mathcal{L}(x) = \ell(x) = \log \prod_{n=1}^{N} p(x_n | \mu, \sigma^2)$

$= \sum_{n=1}^{N} \log p(x_n | \mu, \sigma^2)$

~~$\ln$~~

$\log p(x_n | \mu, \sigma^2) = \log\left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]\right]$

$= \log \frac{1}{\sqrt{2\pi\sigma^2}} + \log \exp\left[\cancel{\exp} \; -\frac{(x-\mu)^2}{2\sigma^2}\right]$

$= \cancel{\ln} \; \cancel{\log(}$

$= \log (2\pi\sigma^2)^{-\frac{1}{2}} - \frac{(x-\mu)^2}{2\sigma^2}$

$= -\frac{\log(2\pi\sigma^2)}{2} - \frac{(x-\mu)^2}{2\sigma^2}$

$= \frac{-1}{2}\left[\log(2\pi\sigma^2) + \frac{(x-\mu)^2}{\sigma^2}\right]$

$\Rightarrow \sum_{n=1}^{N} \log p(x_n | \mu, \sigma^2) = \frac{-1}{2}\sum_{n=1}^{N} \log(2\pi\sigma^2) + \frac{(x-\mu)^2}{\sigma^2} = \ell(x)$

$= -\frac{N\log(2\pi\sigma^2)}{2} - \sum_{n=1}^{N} \frac{(x-\mu)^2}{\sigma^2}$

②

$$\ell(x) = \frac{-N\log(2\pi\sigma^2)}{2} - \sum_{n=1}^{N} \frac{(x_n-\mu)^2}{2\sigma^2}$$

$$\frac{\partial \ell}{\partial \mu} = \frac{\partial}{\partial \mu}\left[\frac{-N\log(2\pi\sigma^2)}{2} - \frac{N}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2\right]$$

$$(x_n-\mu)^2 = x_n^2+\mu^2-2x_n\mu$$

$$\propto \frac{\partial}{\partial \mu}\sum_{n=1}^{N}(x_n-\mu)^2$$

$$= \frac{\partial}{\partial \mu}\sum_{n=1}^{N} x_n^2+\mu^2-2x_n\mu$$

$$= \sum_{n=1}^{N}\underbrace{\frac{\partial}{\partial \mu}\left[x_n^2+\mu^2-2x_n\mu\right]}_{2\mu-2x_n}$$

$$= \sum_{n=1}^{N} 2\mu-2x_n$$

$$\sum_{n=1}^{N}\underbrace{\frac{\partial}{\partial \mu}(x_n-\mu)^2}$$

$$2(x_n-\mu)\frac{\partial}{\partial \mu}(x-\mu)$$

$$= 2(x_n-\mu)(-1)$$

$$= \cancel{-2x_n\mu+2\mu^2}$$

$$\Rightarrow \cancel{\sum_{n=1}^{N} 2\mu^2-2x_n\mu}$$

$$\Rightarrow 2\mu-2x_n$$

$$\sum_{n=1}^{N} 2\mu-2x_n = \cancel{N(2\mu-2} \sum_{n=1}^{N} 2\mu - 2\sum_{n=1}^{N} x_n$$

$$\Rightarrow 2\mu N - 2\sum_{n=1}^{N} x_n$$

$$\frac{\partial \ell}{\partial \mu} \propto 2\mu N - 2\sum_{n=1}^{N} x_n = 0$$

$$2\mu N = 2\sum_{n=1}^{N} x_n$$

$$\boxed{\mu = \frac{1}{N}\sum_{n=1}^{N} x_n}$$

Gaussian MLE 1D

(3)

$$\ell(x) = -\frac{N\log(2\pi\sigma^2)}{2} - \sum_{n=1}^{N}\frac{(x-\mu)^2}{2\sigma^2}$$

$$\frac{\partial \ell}{\partial\sigma} = \frac{\partial}{\partial\sigma}\left[-\frac{N\log(2\pi\sigma^2)}{2} - \sum_{n=1}^{N}\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\propto \frac{\partial}{\partial\sigma}\frac{N}{2}\log(2\pi\sigma^2) - \frac{\partial}{\partial\sigma}\sum_{n=1}^{N}\frac{(x-\mu)^2}{2\sigma^2}$$

$$= \cancel{\frac{\frac{\partial}{\partial\sigma}(2\pi\sigma^2)}{2\pi\sigma^2}}$$

$$= \frac{\frac{\partial}{\partial\sigma}(2\pi\sigma^2)}{2\pi\sigma^2}\frac{N}{2} - \sum_{n=1}^{N}\frac{\partial}{\partial\sigma}\frac{(x_n-\mu)^2}{2}\cancel{2}\underbrace{\sigma^{-2}}_{\partial/\partial\sigma}$$

$$= \underbrace{\frac{\cancel{4}^{2}\pi\cancel{\sigma}}{2\pi\sigma^{\cancel{2}}}}\frac{N}{2} - \sum_{n=1}^{N}\frac{(x_n-\mu)^2}{2}\cancel{2}\left[-2\sigma^{-3}\right]$$

$$\frac{2}{\sigma}\frac{N}{2} = \frac{N}{\sigma}$$

$$= \frac{N}{\sigma} - \sum_{n=1}^{N}\frac{(x_n-\mu)^2}{\sigma^3} = \frac{2}{\sigma} - \frac{1}{\sigma^3}\sum_{n=1}^{N}(x_n-\mu)^2 \propto \frac{\partial\ell}{\partial\sigma}$$

$$\frac{\partial\ell}{\partial\sigma} = 0 \Rightarrow \frac{N}{\sigma} - \frac{1}{\sigma^3}\underbrace{\sum_{n=1}^{N}(x_n-\mu)^2}_{V} = 0$$

$$\frac{N}{\cancel{\sigma}} = \frac{V}{\sigma\sigma^2} \Rightarrow N\cancel{2}\sigma^2 = V, \quad \sigma = \sqrt{\frac{V}{N}}$$

$$\boxed{\sigma = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(x_n-\mu)^2}}$$

Thus it has been proven.